ANALYTIC DETERMINATION OF TEMPERATURE FIELDS ON AN ELECTRONIC COMPUTER

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Computerization of analytical transformations for determination of the thermal state of bodies of complex shape is described.

The analytical expression of heat transfer regularities possesses clarity, informative value, and parametrization properties and permits rapid execution of multifactorial, multivariational computations. Moreover, accurate analytical solutions are tests for numerical methods being created and in this respect are irreplaceable.

Obtaining analytical descriptions is constrained to a significant extent by the complexity of the algorithms being proposed (their execution often remains the lot of the authors).

A method to obtain analytical solutions of a multidimensional nonstationary heat conduction problem having the following form in the desired temperature $T(M, \tau)$

$$C(M, \tau) T_\tau = \operatorname{div}[\lambda(M, \tau) \nabla T] + q_v(M, \tau), \; M(x, y, z) \in \Omega, \; \tau > 0 \tag{1}$$

$$T(M, 0) = T_0(M), \; M \in \Omega, \tag{2}$$

$$T(M, \tau) = T_w(M, \tau), \; M \in S_1, \; \tau > 0, \tag{3_{I}}$$

$$-\lambda(M, \tau) \nabla T \cdot \mathbf{n} = q(M, \tau), \quad M \in S_2, \quad \tau > 0, \tag{3_{II}}$$

$$-\lambda(M, \tau) \nabla T \cdot \mathbf{n} = \alpha(M, \tau)[T(M, \tau) - T_c(\tau)], \quad M \in S_3, \quad \tau > 0 \tag{3_{III}}$$

on an electronic computer is represented in this paper.

Let us represent the desired solution in the form

$$T(M, \tau) = F(M, \tau) + u(M, \tau), \tag{4}$$

where the function $F(M, \tau)$ satisfies the inhomogeneous boundary conditions (3).

Then substituting (4) into (1)-(3), we obtain a boundary value problem with homogeneous boundary conditions

$$C(M, \tau) u_\tau = \operatorname{div}[\lambda(M, \tau) \nabla u] + Q_v(M, \tau), \quad M \in \Omega, \quad \tau > 0, \tag{1'}$$

$$u(M, 0) = u_0(M), \quad M \in \Omega, \tag{2'}$$

$$u(M, \tau) = 0, \quad M \in S_1, \quad \tau > 0, \tag{$3'_{\mathrm{I}}$}$$

$$\nabla u \cdot \mathbf{n} = 0, \quad M \in S_2, \quad \tau > 0, \tag{$3'_{\mathrm{II}}$}$$

$$-\lambda(M, \tau) \nabla u \cdot \mathbf{n} = \alpha(M, \tau) u(M, \tau), \quad M \in S_3, \quad \tau > 0. \tag{$3'_{\mathrm{III}}$}$$

We have $u_0(M) = T_0(M) - F(M, 0)$, $Q_v(M, \tau) = q_v(M, \tau) + \operatorname{div}[\lambda(M, \tau) \nabla F(M, \tau)] - C(M, \tau) F_\tau(M, \tau)$ in (1')-(3'). We will seek the function $u(M, \tau)$ in the form

$$u(M, \tau) = \sum_{i=1}^{n} \chi_i(M, \tau) \psi_i(\tau). \tag{5}$$

Here $\psi_i(\tau)$ are unknown functions of the time, and $\chi_i(M, \tau)$ is a complete system in Ω of coordinate functions selected to satisfy a priori conditions (3') and dependent on τ in a known manner since it includes $\lambda(M, \tau)$ and $\alpha(M, \tau)$ for boundary conditions of the third kind.

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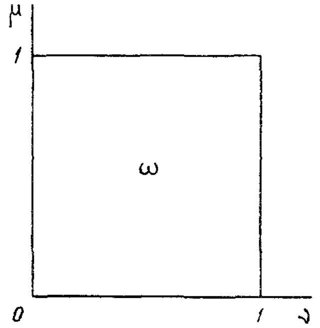

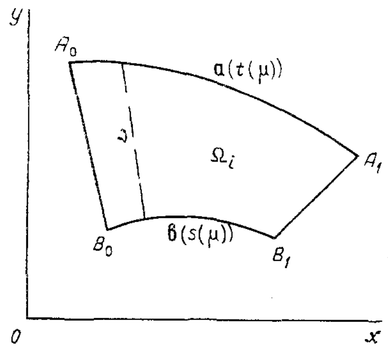

Fig. 1 Fig. 2

Fig. 1. The unit square.

Fig. 2. A universal subdomain of a plane domain.

It is shown in [1, 2] that when utilizing a variational description with a convolution function we have a Cauchy problem in $\psi_i(\tau)$

$$\sum_{i=1}^{n} A_{ji}(\tau)\,\psi_i'(\tau) = \sum_{i=1}^{n} a_{ji}(\tau)\,\psi_i(\tau) + A_j(\tau),$$
$$\sum_{i=1}^{n} A_{ji}(0)\,\psi_i(0) = \int_{\Omega} C(M,\ 0)\,u_0(M)\,\chi_j\,d\Omega,\quad j=1,\ \ldots,\ n, \tag{6}$$

where

$$a_{ji}(\tau) = \int_{\Omega} \operatorname{div}[\lambda(M,\ \tau)\,\nabla\chi_i]\,\chi_j d\Omega - \int_{\Omega} C(M,\ \tau)\,\chi'_{i,\tau}\,\chi_j d\Omega;$$
$$A_{ji}(\tau) = \int_{\Omega} C(M,\ \tau)\,\chi_i\chi_j d\Omega;\quad A_j(\tau) = \int_{\Omega} Q_v(M,\ \tau)\,\chi_j d\Omega. \tag{7}$$

The first approximation to the solution of the problem (1')-(3') in the form

$$u_1(M,\ \tau) = \psi_1(\tau)\,\chi_1(M,\ \tau) \tag{8}$$

contains $\psi_1(\tau)$ thus:

$$\psi_1(\tau) = \left\{\int_0^{\tau} [A_1(\tilde\tau)/A_{11}(\tilde\tau)]\exp\left\{-\int_0^{\tilde\tau} [a_{11}(\theta)/A_{11}(\theta)]\,d\theta\right\} d\tilde\tau + \psi_1(0)\right\}\exp\left\{\int_0^{\tau} [a_{11}(\tilde\tau)/A_{11}(\tilde\tau)]\,d\tilde\tau\right\}, \tag{9}$$

where

$$\psi_1(0) = \int_{\Omega} C(M,\ 0)\,u_0(M)\,\chi_1 d\Omega/A_{11}(0).$$

It is easy to find the analytic second approximation

$$u_2(M,\ \tau) = \psi_1(\tau)\,\chi_1(M,\ \tau) + \psi_2(\tau)\,\chi_2(M,\ \tau) \tag{10}$$

for a_{ji} = const, A_{ji} = const (this case corresponds to the dependence of C, λ, and α on only the coordinates) and is possible just for certain dependences of the coefficients a_{ji} and A_{ji} on τ.

It follows from the above that the main step in the method developed to obtain the first and second analytic expressions to the solution of nonstationary heat conduction problems is to determine the analytic dependences for the functions $a_{ji}(\tau)$, $A_{ji}(\tau)$, $A_j(\tau)$ of the system of equations (6) by means of (7). These same dependences must be set up to save computation time and before the numerical solution of the system (6) when searching for the n-th (n > 2) nonanalytic approximation to the solution of the problem (1)-(3).

It seemed expediate to us to transfer the analytic calculations of the integrals (7) over to an electronic computer when utilizing the results in the development of the pro-

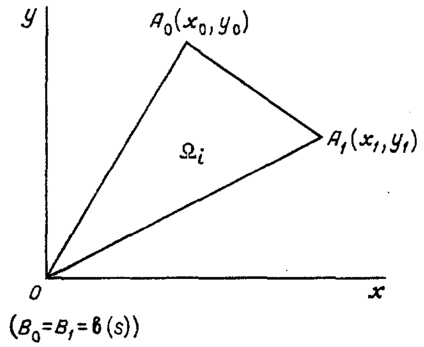

Fig. 3

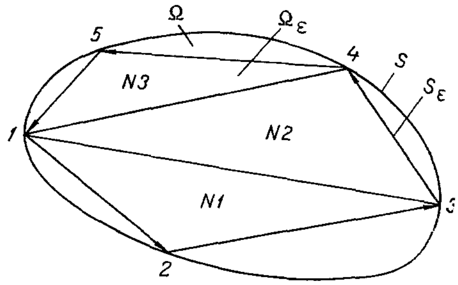

Fig. 4

Fig. 3. Triangle subdomain.

Fig. 4. The plane domain Ω and its polygonal equivalent Ω_ε.

gramming systems for the analytic transformations [3]. To do this a packet of applied programs was produced for the analytic integration of polynomials in plane domains.

Vectors with polynomial components called vector-polynomials [4] are a convenient object for the analytic transformations in the evaluation of the multiple integrals in (7). As a rule it is necessary to use several vector-polynomials mutually consistent at the nodal points of the boundary, i.e., a polynomial spline, for the description of the boundary of the domain Ω. Let us also assume that the integrands are polynomials in the variables of integration, the coordinates x, y and the time τ. Therefore, the functions $C(x, y, \tau)$, $\lambda(x, y, \tau)$, $q_v(x, y, \tau)$, $\tilde{T}_w(x, y, \tau)$, $q(x\ y, \tau)$, $T_c(\tau)$, $\alpha(x, y, \tau)$ known in the formulation of the problem, and the coordinate functions $\chi_i(x, y, \tau)$, $\chi_j(x, y, \tau)$ are given by polynomials.

It is specified that numbers, symbols, or their combinations can be coefficients of the polynomials. A name is conferred on each polynomial in the program and upon insertion in the electronic computer turns out to be a quantity of the variables and its terms (monomials) used in them, whose introduction is performed sequentially as x^0, y^0, x, y, x^2, xy, y^2, x^3, x^2y, xy^2, y^3 etc. If some monomial is absent from the polynomial, then the zeroth coefficient associated with it is inserted.

Then the procedure for the analytic calculation of a double integral over a plane domain Ω will consist of two stages:

1) the partition of Ω into a set of elementary subdomains Ω_i and the corresponding representation of the multiple integral as

$$\int_{\Omega} p(\tilde{\Omega})\, d\tilde{\Omega} = \sum_{i=1}^{k} \int_{\Omega_i} p(\tilde{\Omega})\, d\tilde{\Omega}, \tag{11}$$

where $p(\tilde{\Omega})$ is a polynomial defined in $\Omega = \bigcup_{i=1}^{k} \Omega_i$;

2) the mapping of Ω_i in the unit square (Fig. 1).

A universal subdomain of Ω_i in the form of a curvilinear trapezoid whose opposite sides are Jordan arcs is shown in Fig. 2. On this figure A_0B_0, A_1B_1 are line segments, A_0A_1, B_0B_1 are Jordan arcs, each of which is described by vector-polynomials with parameters t and s, respectively

$$\mathbf{A}_0\mathbf{A}_1 = \mathbf{a}(t) = \mathbf{a}_0 + \mathbf{a}_1 t + \ldots + \mathbf{a}_n t^n, \quad \mathbf{B}_0\mathbf{B}_1 = \mathbf{b}(s) = \mathbf{b}_0 + \mathbf{b}_1 s + \ldots + \mathbf{b}_n s^n. \tag{12}$$

We evidently obtain subdomains in the form of triangles with one or two curvilinear sides when B_0 coincides with B_1 or B_0 with A_0 (or B_1 with A_1). The curvilinear trapezoid goes over into a segment during the simultaneous coincidence of B_0 with A_0 and B_1 with A_1. Moreover, by partitioning the triangle with three curvilinear sides into two parts, there results a triangle with two curvilinear sides.

TABLE 1. Results of an Analytic (top row) and Numerical (bottom row) Finite Element Computation of the Temperature Field in a Triangular Prism

Time τ	Temperature at points M(x, y) of the prism transverse section							
	$x=1,00$; $y=-0,45$	$x=0,85$; $y=-0,35$	$x=0,825$; $y=-0,175$	$x=0,825$; $y=-0.075$	$x=0,35$; $y=0,00$	$x=0,70$; $y=0,075$	$x=0,70$; $y=0,20$	$x=1,00$; $y=0,55$
0,01	1,0000 1,0000	0,8029 0,8105	0,4098 0,3632	0,2618 0,2400	0,4584 0,3783	—0,0515 0,0375	—0,0101 0,0069	1,0000 1,0000
0,02	1,0000 1,0000	0,8652 0,9180	0,5962 0,6345	0,4948 0,5010	0,6294 0,6485	0,2804 0,2541	0,3088 0,3060	1,0000 1,0000
0,05	1,0000 1,0000	0,9568 0,9777	0,8706 0,8868	0,8382 0,8358	0,8816 0,8903	0,7695 0,7393	0,7786 0,7632	1,0000 1,0000
0,07	1,0000 1,0000	0,9798 0,9893	0,9394 0,9454	0,9242 0,9208	0,9444 0,9479	0,8920 0,8782	0,8693 0,8652	1,0000 1,0000
0,10	1,0000 1,0000	0,9935 0,9964	0,9606 0,9817	0,9758 0,9735	0,9822 0,9825	0,9654 0,9580	0,9668 0,9620	1,0000 1,0000

It must be noted that the points $\mathbf{A}_0=\mathbf{a}(t_0)$, $\mathbf{A}_1=\mathbf{a}(t_1)$, $\mathbf{B}_0=\mathbf{b}(s_0)$, $\mathbf{B}_1=\mathbf{b}(s_1)$ can be considered vectors having values of the abscissas and ordinates as projections on the coordinate axes.

The following universal method of parametrization of the subdomains Ω_i displayed in Fig. 2 is proposed below by following [4].

We go from the parameters t and s in (12) over to the parameter $\mu\in[0, 1]$ by means of the rule

$$t=(1-\mu)\,t_0+\mu t_1,\quad s=(1-\mu)\,s_0+\mu s_1 \tag{13}$$

and we join the points $\mathbf{A}(\mu)=\mathbf{a}(t(\mu))$ and $\mathbf{B}(\mu)=\mathbf{b}(s(\mu))$ by a line segment with parameter $\nu\in[0, 1]$. Then the subdomain Ω_i will be parametrized as follows

$$\boldsymbol{\omega}(\mu,\ \nu)=(1-\nu)\,\mathbf{A}(\mu)+\nu\mathbf{B}(\mu),\quad \mu,\ \nu\in[0,\ 1]. \tag{14}$$

The Jacobian for the passage from Ω_i to $\omega(\mu, \nu)$ here equals

$$J=\left|\frac{D(\omega_1,\ \omega_2)}{D(\mu,\ \nu)}\right|=(1-\nu)\det(\mathbf{B}-\mathbf{A},\ \mathbf{A}')+\nu\det(\mathbf{B}-\mathbf{A},\ \mathbf{B}'). \tag{15}$$

The determinants in the right side of (14) are polynomials of the arguments μ and $1-\mu$.

Each integral in the right side of (11) can now be rewritten in the form

$$\int_{\tilde{\Omega}_i} p(\tilde{\Omega})\,d\tilde{\Omega}=\int_{\mu=0}^{1}\int_{\nu=0}^{1} p(\boldsymbol{\omega}(\mu,\ \nu))\,J(\mu,\ \nu)\,d\nu d\mu, \tag{16}$$

where $p(\omega)J$ is a polynomial in the variables μ, $1-\mu$, ν, $1-\nu$. Consequently, the integral (16) is easily evaluated analytically. A significant saving in machine time is achieved if the formula

$$\int_0^1 \nu^i(1-\nu)^j\,d\nu=i!j!/(i+j+1)! \tag{17}$$

is used here.

The simplest example for the application of the approach developed here is the parametrization of the subdomain Ω_i that is a triangle when the points B_0 and B_1 coincide with the origin (Fig. 3).

The lines $\mathbf{A}_0\mathbf{A}_1$ and $\mathbf{B}_0\mathbf{B}_1$ are described by the vector-polynomials

$$\mathbf{A}_0\mathbf{A}_1=\mathbf{a}(t)=(1-t)\,\mathbf{A}_0+t\mathbf{A}_1,\quad \mathbf{B}_0\mathbf{B}_1=\mathbf{b}(s)=0, \tag{18}$$

in which $t_0 = 0$, $t_1 = 1$, so that when going over from the parameters t and s to the parameter $\mu\in[0, 1]$ we have

$AM(1,1) = -8/15\ A2\ H^5A^7 - 8/45\ A2\ H^7A^5 - 12/35\ B2\ H^7$

$A^7 - 4/63\ B2\ H^9A^5 - 4/63\ C2\ H^5A^9 - 4/105\ C2\ H^7A^7$

$-16/315\ E2\ H^6A^9 - 8/315\ E2\ H^8A^7$

$A(1,1) = 8/315\ A1\ H^7A^7 + 8/675\ B1\ H^9A^7 + 8/4725\ C1\ H^7$

$A^9 + 64/7425\ F1\ H^{10}A^7 + 64/51975\ E1\ H^8A^9$

$A(1) = 4\ TP\ A2\ H^2A^4 + 4/3\ TP\ A2\ H^4A^2 + 46/15\ TP\ B2\ H^4$

$A^4 + 2/5\ TP\ B2\ H^6A^2 + 14/15\ TP\ C2\ H^2A^6 + 2/5\ TP\ C2\ H^4$

$A^4 + 4/5\ TP\ E2\ H^3A^6 + 4/15\ TP\ E2\ H^5A^4$

$PS10(1) = 2/15\ TO\ A1\ H^4A^4 + 4/63\ TO\ B1\ H^6A^4 + 4/315\ TO$

$C1\ H^4A^6 + 1/21\ TO\ F1\ H^7A^4 + 1/105\ TO\ E1\ H^5A^6$

Fig. 5. Components of the analytic solution of a heat conduction problem for an isosceles triangle (Labtam-3015/16 computer ADC printer).

$$\mathbf{A}(\mu) = \mathbf{a}(t(\mu)) = \mathbf{a}(\mu) = (1-\mu)\mathbf{A}_0 + \mu\mathbf{A}_1, \quad \mathbf{B}(\mu) = \mathbf{b}(s(\mu)) = \mathbf{B}_0(0). \tag{19}$$

The subdomain Ω_i turns out to be parametrized in the following manner:

$$\boldsymbol{\omega}(\mu, \nu) = (1-\nu)[(1-\mu)\mathbf{A}_0 + \mu\mathbf{A}_1] + \nu\mathbf{B}_0 = (1-\nu)[(1-\mu)\mathbf{A}_0 + \mu\mathbf{A}_1]$$

and the Jacobian of the transition from Ω_i to $\omega(\mu, \nu)$ is

$$J = |(1-\nu)(\mathbf{A}_1 - \mathbf{A}_0) \quad (\mu - 1)\mathbf{A}_0 - \mu\mathbf{A}_1| = (1-\nu)\begin{vmatrix} x_1 - x_0 & -x_0 \\ y_1 - y_0 & -y_0 \end{vmatrix} = (1-\nu)(x_0y_1 - x_1y_0). \tag{20}$$

The second cofactor in the right side of (20) agrees with twice the area of the triangle $A_0A_1B_0$.

It is also clear that x and y must be replaced in the integrand of the right side of (16) according to the rule

$$x = \omega_x = (1-\nu)[(1-\mu)x_0 + \mu x_1], \quad y = \omega_y = (1-\nu)[(1-\mu)y_0 + \mu y_1].$$

A program for the analytic determination of the functions $a_{ji}(\tau)$, $A_{ji}(\tau)$, $A_j(\tau)$, $\psi_i(0)$, called "Integration in a polygonal domain," was produced on the basis of the results obtained, in the symbolic programming language PL/1-FORMAC [3].* Integration in a plane simply-connected domain Ω bounded by a curve S was replaced by integration in the polygonal domain Ω_ε, i.e., in a polygon. Here Ω_ε is partitioned into triangular elements (Fig. 4). The explicit parts of the polygon boundary equations (broken line linkages) are determined on the electronic computer from the formula

$$w_i = (y_{i+1} - y_i)(x - x_i) - (x_{i+1} - x_i)(y - y_i), \quad i = 1, \ldots, N. \tag{21}$$

Afterwards the coordinate functions χ_i, χ_j are evaluated or inserted. For instance, we have for the case of homogeneous boundary conditions of the first kind

$$\chi_1 = (-1)^N \prod_{i=1}^{N} w_i, \ \chi_2 = x\chi_1, \ \chi_3 = y\chi_1, \ \chi_4 = x^2\chi_1, \ \ldots \tag{22}$$

A printout of the data inserted in the computer storage and the results of its computation of the functions a_{ji}, A_{ji}, A_j, $\psi_i(0)$ inserted in (9) to determine the temperature field in a first approximation in an unbounded prism having an isosceles triangle with vertex coordinates $M_1(0, 0)$, $M_2(h, -a)$, $M_3(h, a)$ is presented in Fig. 5 to demonstrate operation of the program. A constant temperature T_w = const was given on the prism surface and we assumed a uniform initial temperature distribution therein T_0 = const in the absence of

*The program was compiled jointly with I. M. Bakirov.

sources of bulk heat liberation. The thermophysical characteristics of the body material were selected thus:

$$C(x,\ y) = a_1 + b_1x^2 + c_1y^2 + d_1x^2y + e_1xy^2 + f_1x^3,\quad \lambda(x,\ y) = a_2 + b_2x^2 + c_2y^2 + d_2x^2y + e_2xy^2. \tag{23}$$

The following identifiers (Fig. 5) were used in the program

$$a_{11} = AM(1,\ 1),\quad A_{11} = A(1,\ 1),\quad A_1 = A(1),\quad \psi_1(0)\cdot A_{11}(0) = PSI0\,(1),$$

$$T_w = TP,\quad T_0 = T0,\quad h = H,\quad a = A,\quad a_1 = A1,\quad a_2 = A2,\quad b_1 = B1,$$

$$b_2 = B2,\quad c_1 = C1,\quad c_2 = C2,\quad d_1 = D1,\quad d_2 = D2,\quad e_1 = E1,\quad e_2 = E2,\quad f_1 = F1.$$

The program created was then used also to construct the first approximation to the determination of the temperature field in an unbounded prism of triangular section formed by the planes $x = 1$, $y = k_1x$, $y = k_2x$ under the conditions of the preceding problem when the section vertices have the coordinates $M_1(0,\ 0)$, $M_2(1,\ k_1)$, $M_3(1,\ k_2)$.

It is expedient to note that obtaining the analytic first approximation in the identifiers require 78 and 540 sec ES-1060 computer machine time, respectively, for the triangles under consideration in the prism section. The time to perform the computations was reduced sharply when using numbers as polynomial coefficients. Thus 18 sec ES-1060 computer machine time was expended in determining $T(x,\ y,\ \tau)$ in a nonisosceles prism with $k_1 = -0.5$, $k_2 = 0.8$, $a_1 = a_2 = 1$, $b_1 = b_2 = c_1 = c_2 = d_1 = d_2 = e_1 = e_2 = f_1 = 0$. To obtain the same results presented in the table in the case of using a well-known finite element method (FEM) 1400 sec was expended on the same computer. Moreover, when using the FEM very much time was required additionally in executing routine work and work to prepare the original digital information associated with the potential introduction of errors.

NOTATION

Ω and S, geometric domain and its boundary surface with the parts S_1, S_2, and S_3; x, y, z, and τ, coordinates and time; $C(M,\ \tau)$ and $\lambda(M,\ \tau)$, bulk specific heat and heat conduction coefficient; q_V, power of the bulk heat liberation sources; $T_W(M,\ \tau)$ and $T_C(\tau)$, surface temperature on S_1 and of surrounding medium at S_3; $q(M,\ \tau)$, heat flux density on S_2; and $\alpha(M,\ \tau)$, coefficient of convective heat elimination to S_3.

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